Some homological properties of an amalgamated duplication of a ring along an ideal

Mohamed Chhiti and Najib Mahdou

Department of Mathematics
Faculty of Science and Technology of Fez, P.O. Box 2202
University S. M. Ben Abdellah, Fez, Morocco
chhiti.med@hotmail.com
mahdou@hotmail.com

Abstract. In this work, we investigate the transfer of some homological properties from a ring R to his amalgamated duplication along some ideal I of R, and then generate new and original families of rings with these properties.

Key Words. Amalgamated duplication of a ring along an ideal, Von Neumann regular ring, Perfect ring, (n,d)-ring and weak(n,d)-ring, Coherent ring, Uniformly coherent ring.

1 Introduction

Let R be a commutative ring with unit element 1 and let I be a proper ideal of R. The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i) \mid r \in R, i \in I\}$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by M D'Anna and M Fontana [3]. Also, M. D'Anna and M.

Fontana, in [2], have considered the case of the amalgamated duplication of a ring ,in not necessarily Noetherian setting, along a multiplicative -canonical ideal in the sense Heinzer-Huckaba-Papick [10]. In [1], M. D'Anna has studied some properties of $R \bowtie I$, in order to construct reduced Gorenstein rings assosiated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, H.R Maimani and S Yassemi, in [15], have studied the diameter and girth of the zero- divisor of the ring $R \bowtie I$. For instance, see [1, 2, 3, 15].

Let M be an R-module, the idealization $R \propto M$ (also called the trivial extention), introduced by Nagata in 1956 (cf [16]) is defined as the R-module $R \oplus M$ with multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. For instance, see [8, 9, 11, 12].

When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the idealization $R \propto I$. One main difference of this construction, with respect to idealization is that the ring $R \bowtie I$ can be a reduced ring (and, in fact, it is always reduced if R is a domain).

For two rings $A \subset B$, we say that A is a module retract (or a subring retract) of B if there exists an A-module homomorphism $\varphi : B \to A$ such that $\varphi \mid_A = id \mid_A$. φ is called a module retraction map. If such a map φ exists, B contains A as an A-module direct summand. We can easily show that R is a module retract of $R \bowtie I$, where the module retraction map φ is defined by $\varphi(r, r + i) = r$.

In this paper, we study the transfer of some homological properties from a ring R to a ring $R \bowtie I$. Specially, in section 2, we prove that $R \bowtie I$ is a Von Neumann regular ring (resp., a perfect ring) if and only if so is R. Also, we prove that $gldim(R \bowtie I) = \infty$ if R is a domain and I is a principal ideal of R. In section 3, we study the coherence of $R \bowtie I$. More precisely, we prove that if R is a coherent ring and I is a finitely generated ideal of R, then $R \bowtie I$ is coherent. And if I contains a regular element, we prove the converse.

Recall that if R is a ring and M is an R-module, as usual we use $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the usual projective and flat dimensions of M, respectively. The classical global and weak dimension of R are respectively denoted by $\mathrm{glim}(R)$ and $\mathrm{wdim}(R)$. Also, the Krull dimension of R is denoted by $dim(R)$.

2 Transfer of some homological properties from a ring R to a ring $R \bowtie I$

Let R be a commutative ring with identity element 1 and let I be an ideal of R. We define $R \bowtie I = \{(r, s)/r, s \in R, s - r \in I\}$. It is easy to check that $R \bowtie I$ is a subring with unit element $(1, 1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r + i)/r \in R, i \in I\}$.

It is easy to see that, if $\pi_i (i = 1, 2)$ are the projections of $R \times R$ on R, then

$\pi_i(R \bowtie I) = R$ and hence if $O_i = ker(\pi_i \backslash R \bowtie I)$. Then $R \bowtie I/O_i \cong R$. Moreover $O_1 = \{(0,i), i \in I\}$, $O_2 = \{(i,0), i \in I\}$ and $O_1 \cap O_2 = (0)$.

We begin by studying the transfer of Von Neumann regular property.

Theorem 2.1 *Let R be a commutative ring and let I be a proper ideal of R. Then R is a Von Neumann regular ring if and only if $R \bowtie I$ is a Von Neumann regular ring.*

The proof will use the following Lemma.

Lemma 2.2 *[3, Theorem 3.5]*

1. *Let R be a commutative ring and let I be an ideal of R. Let P be a prime ideal of R and set:*

$$\begin{aligned} P_0 &= \{(p, p+i)/p \in P, i \in I \cap P\} \\ P_1 &= \{(p, p+i)/p \in P, i \in I\} \\ \text{and} \qquad P_2 &= \{(p+i, p)/p \in P, i \in I\} \end{aligned}$$

   - *If $I \subseteq P$, then $P_0 = P_1 = P_2$ is a prime ideal of $R \bowtie I$ and it is the unique prime ideal of $R \bowtie I$ lying over P.*
   - *If $I \nsubseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$ and P_1 and P_2 are the only prime ideals of $R \bowtie I$ lying over P.*

2. *Let Q be a prime ideal of $R \bowtie I$ and let $O_1 = \{(0,i)/i \in I\}$. Two cases are possible: either $Q \nsupseteq O_1$ or $Q \supseteq O_1$.*

   a- *If $Q \nsupseteq O_1$, then there exists a unique prime ideal P of R $(I \nsubseteq P)$such that*

$$Q = P_2 = \{(p+i, p)/p \in P, i \in I\}$$

   b- *If $Q \supseteq O_1$, then there exists a unique prime ideal P of R such that*

$$Q = P_1 = \{(p, p+i)/p \in P, i \in I\}$$

Proof of Theorem 2.1. Assume that R is a Von Neumann regular ring. Then R is reduced and so $R \bowtie I$ is reduced by [3, Theorem 3.5 (a)(vi)]. It remains to show that $dim(R \bowtie I) = 0$ by [9, Remark, p. 5].

Let Q be a prime ideal of $R \bowtie I$. If $P = Q \cap R$, then necessarily $Q \in \{P_1, P_2\}$ (by Lemma 2.2(2)). But P is a maximal ideal of R since R is a Von Neumann regular

ring. Then P_1 and P_2 are maximal ideals of $R \bowtie I$ (by [3, Theorem 3.5 (a)(vi)]). Hence, Q is a maximal ideal of $R \bowtie I$, as desired.

Conversely, assume that $R \bowtie I$ is a Von Neumann regular ring. By [3, Theorem 3.5 (a)(vi)], R is reduced. Let P be a prime ideal of R. By Lemma 2.2(1), $P \bowtie I = \{(p, p+i)/p \in P, i \in I\}$ is a prime ideal of $R \bowtie I$. From [9, page 7] we get $P \bowtie I$ is a maximal ideal of $R \bowtie I$ and hence P is a maximal ideal of R. Therefore, $dim(R) = 0$ and so R is a Von Neumann regular ring . ∎

Corollary 2.3 *Let R be a commutative ring and let I be a proper ideal of R. Then R is a semisimple ring if and only if $R \bowtie I$ is a semisimple ring.*

Proof. Assume that R be a semisimple ring. Then R is a Noetherian Von Neumann regular ring. By Theorem 2.1, $R \bowtie I$ is a Von Neumann regular ring and by [3, Corollary 3-3], $R \bowtie I$ is Noetherian. Therefore $R \bowtie I$ is semisimple.

Conversely, assume that $R \bowtie I$ is semisimple. Then $R \bowtie I$ is a Noetherian Von Neumann regular ring and so R is a Von Neumann regular ring (by Theorem 2.1) and Noetherian (by [3, Corollary 3-3]). Hence, R is semisimple. ∎

A ring R is called a stably coherent ring if for every positive integer n, the polynomial ring in n variables over R is a coherent ring. Recall that a ring R is is called a coherent ring if every finitely generated ideal of R is finitely presented.

Corollary 2.4 *Let R be a commutative ring and let I be a proper ideal of R. If R is a Von Neumann regular ring, then $R \bowtie I$ is a stably coherent ring.*

Proof. By Theorem 2.1 and [8, Theorem 7.3.1] ∎

Now, we are able to construct a new class of non-Noetherian Von Neumann regular rings.

Example 2.5 *Let R be a non-Noetherian Von Neumann regular ring, and let I be a proper ideal of R. Then, $R \bowtie I$ is a non-Noetherian Von Neumann regular ring, by [3, Corollary 3-3] and Theorem 2.1.*

We recall that a ring R is called a perfect ring if every flat R-module is a projective R-module (see [4]). Secondly, we study the transfer of perfect property.

Theorem 2.6 *Let R be a commutative ring and let I be a proper ideal of R. Then R is a perfect ring if and only if $R \bowtie I$ is a perfect ring.*

Before proving this Theorem , we need the following Lemmas .

Lemma 2.7 *([13, Lemma 2.5.(2)])*
Let $(R_i)_{i=1,2}$ *be a family of rings and* E_i *be an* R_i*-module for* $i = 1, 2$. *Then* $\mathrm{pd}_{R_1\times R_2}(E_1 \times E_2)$ =$\sup\{\mathrm{pd}_{R_1}(E_1), \mathrm{pd}_{R_2}(E_2)\}$.

Lemma 2.8 *Let* $(R_i)_{i=1,2}$ *be a family of rings and* E_i *be an* R_i*-module for* $i = 1, 2$. *Then* $\mathrm{fd}_{R_1\times R_2}(E_1 \times E_2)$ =$\sup\{\mathrm{fd}_{R_1}(E_1), \mathrm{fd}_{R_2}(E_2)\}$.

Proof. The proof is analogous to the proof of Lemma 2.7.

Lemma 2.9 *Let* $(R_i)_{i=1,...,m}$ *be a family of rings. Then* $\prod_{i=1}^{m} R_i$ *is a perfect ring if and only if* R_i *is a perfect ring for each* $i = 1, ..., m$.

Proof. The proof is done by induction on m and it suffices to check it for $m = 2$. Let R_1 and R_2 be two rings such that $R_1 \times R_2$ is perfect. Let E_1 be a flat R_1-module and let E_2 be a flat R_2-module. By Lemma 2.8, $E_1 \times E_2$ is a flat $R_1 \times R_2$ module and so it is a projective $R_1 \times R_2$ module since $R_1 \times R_2$ is a perfect ring. Hence, E_1 is a projective R_1-module and E_2 is a projective R_2-module by Lemma 2.7; that means that R_1 and R_2 are perfect rings.

Conversely, assume that R_1 and R_2 are two perfect rings. Let $E_1 \times E_2$ be a flat $R_1 \times R_2$-module where E_i is an R_i-module for each $i = 1, 2$. By Lemma 2.8, E_1 is a flat R_1-module and let E_2 is a flat R_2-module; so E_1 is a projective R_1-module and E_2 is a projective R_2-module. Therefore $E_1 \times E_2$ is a projective $R_1 \times R_2$ by Lemma 2.7, this means that $R_1 \times R_2$ is a perfect rings. ■

Lemma 2.10 *Let* R *be a commutative ring and let* I *be a proper ideal of* R. *Then:*

1. *An* $(R \bowtie I)$*- module* M *is projective if and only if* $M \otimes_{R\bowtie I} (R \times R)$ *is a projective* $(R \times R)$ *module and* M/O_1M *is a projective* R*- module.*

2. *An* $(R \bowtie I)$*-module* M *is flat if and only if* $M \otimes_{R\bowtie I} (R\times R)$ *is a flat* $(R\times R)$*-module and* M/O_1M *is a flat* R*- module.*

Proof. Note that $R \bowtie I$ is a subring of $R \times R$ and O_1 is a common ideal of $R \bowtie I$ and $R\times R$ by [3, proposition 3-1]. The result follows from [8, theorem 5-1-1]. ■

Proof of Theorem 2.7 Assume that R is a perfect ring and let M be a flat $(R \bowtie I)$-module. By Lemma 2.10(2), $M \otimes_{R \bowtie I} (R \times R)$ is a flat $(R \times R)$-module and M/O_1M is a flat R- module. Then $M \otimes_{R \bowtie I} R \times R$ is a projective $R \times R$- module (since $R \times R$ is perfect by Lemma 2.9), and M/O_1M is a projective R-module since R is perfect. By Lemma 2.10(1), M is a projective $(R \bowtie I)$-module and so $R \bowtie I$ is a perfect ring.

Conversely, assume that $R \bowtie I$ is a perfect ring and let E be a flat R- module. Then $E \otimes_R (R \bowtie I)$ is a flat $(R \bowtie I)$-module and so it is a projective $(R \bowtie I)$-module since $R \bowtie I$ is a perfect ring. In addition, for any R- module M and any $n \geq 1$ we have:

$$Ext^n_R(E, M \otimes_R R \bowtie I) \cong Ext^n_R(E \otimes_R R \bowtie I, M \otimes_R R \bowtie I)$$

(see [6, page 118]) and then $Ext^n_R(E, M \otimes_R R \bowtie I) = 0$. As we note that M is a direct summand of $M \otimes_R R \bowtie I$ since R is a module retract of $R \bowtie I$, $Ext^n_R(E, M) = 0$ for all $n \geq 1$ and all R- module M. This means that E is a projective R- module and so R is a perfect ring. ■

We say that a ring R is Steinitz if any linearly independent subset of a free R-module F can be extended to a basis of F by adjoining element of a given basis. In [7, proposition 5.4], Cox and Pendleton showed that Steinitz rings are precisely the perfect local rings.

By the above Theorem and since $R \bowtie I$ is local if and only if R is local, we obtain:

Corollary 2.11 *Let R be a commutative ring and let I be a proper ideal of R. Then R is a Steinitz ring if and only if $R \bowtie I$ is a Steinitz ring.*

Example 2.12 *Let $R = K[X]/(X^2)$ where K is a field and X an indetrminate. Then $(K[X]/(X^2)) \bowtie (\overline{X})$ is a Steinitz ring.*

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence $F_n \to F_{n-1} \to ... \to F_0 \to E \to 0$ in which each F_i is a finitely generated free R-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented.

Given nonnegative integers n and d, a ring R is called an (n, d)-ring if every n-presented R-module has projective dimension $\leq d$; and R is called a weak (n, d)-ring if every n-presented cyclic R-module has projective dimension $\leq d$ (equivalently, if every $(n-1)$-presented ideal of R has projective dimension $\leq d-1$). For instance, the $(0, 1)$-domains are the Dedekind domains, the $(1, 1)$-domains are the Prüfer domains, and the $(1, 0)$-rings are the von Neumann regular rings. See for instance ([5], [11], [12], [13], [14]).

Now, we give a wide class of rings which are not a weak (n, d)-ring (and so not an (n, d)-ring) for each positive integers n and d.

Theorem 2.13 *Let R be an integral domain and let $I(\neq 0)$ be a principal ideal of R. Then R is not a weak (n,d)-ring (and so not an (n,d)-ring) for each positive integers n and d. In particular, $wdim(R \bowtie I) = gldim(R \bowtie I) = \infty$.*

Before proving this Theorem, we need the following Lemma.

Lemma 2.14 *Let R be a commutative ring and let $I(\neq 0)$ be a principal ideal of R, then $O_1 = \{(0,i), i \in I\}$ and $O_2 = \{(i,0), i \in I\}$ are principal ideals of $R \bowtie I$.*

Proof. Let $(0,i)$ be an element of O_1. Since I is a principal ideal of R, then there exists $a \in I$ such that $I = Ra$ and so $(0,i) = (0,ra) = (r+j,r)(0,a)$ for some $r \in R$ and for all $j \in I$. Hence, O_1 is a principal ideal of $R \bowtie I$ generated by $(0,a)$. Also, O_2 is a principal ideal generated by $(a,0)$ by the same argument, as desired. ■

Proof of Theorem 2.13. Let $a \in I$ such that $I = Ra$. By lemma 2.14, O_1 and O_2 are principal ideals of $R \bowtie I$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(1) \qquad 0 \to ker(u) \to R \bowtie I \xrightarrow{u} O_1 \to 0$$

where $u(r, r+i) = (r, r+i)(0,a) = (0,(r+i)a)$. Then, $ker(u) = \{(r,0) \in R \bowtie I / r \in I\} = O_2$. Consider the short exact sequence of $R \bowtie I$-modules:

$$(2) \qquad 0 \to ker(v) \to R \bowtie I \xrightarrow{v} O_2 \to 0$$

where $v(r, r+i) = (r, r+i)(a,0) = (ra,0)$. Then, $ker(v) = \{(0,i) \in R \bowtie I / i \in I\} = O_1$. Therefore, O_1 (resp., O_2) is m-presented for each positive integer m by the above two exact sequences. It remains to show that $pd_{R\bowtie I}(O_1) = \infty$ (or $pd_{R\bowtie I}(O_2) = \infty$).

We claim that O_1 and O_2 are not projective. Deny. Then O_1 is projective and so the short exact sequence (1) splits. Then O_2 is generated by an idempotent element $(x,0)$, such that $x(\neq 0) \in I$. Hence, $(x,0)^2 = (x,0)(x,0) = (x^2,0) = (x,0)$, then $x^2 = x$, and so $x = 1$ or $x = 0$, a contradiction (since $x \in I$ and $x \neq 0$). Therefore, O_1 is not projective. Similar arguments show that O_2 is not projective. A combination of (1) and (2) yields $pd_{R\bowtie I}(O_1) = pd_{R\bowtie I}(O_2) + 1$ and $pd_{R\bowtie I}(O_2) = pd_{R\bowtie I}(O_1) + 1$ then, $pd_{R\bowtie I}(O_1) = pd_{R\bowtie I}(O_2)+1+1 = pd_{R\bowtie I}(O_1)+2$. Consequently, the projective dimension of O_1 (resp., O_1) has to be infinite, as desired. ■

If R is a principal domain, we obtain:

Corollary 2.15 *Let R be a principal domain and let I be a proper ideal of R. Then R is not a weak (n,d)-ring (and so not an (n,d)-ring) for each positive integers n and d. In particular, $wdim(R \bowtie I) = gldim(R \bowtie I) = \infty$.*

3 The coherence of $R \bowtie I$

An R- module M is called a coherent R module, if it is a finitely generated and every finitely generated submodule of M is finitely presented.

A ring R is called a coherent ring if it is a coherent module over itself, that is, if every finitely generated ideal of R is finitely presented, equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R (by [8, Theorem 2.2.3]). Examples of coherent rings are Noetherian rings, Boolean algebras, Von Neumann regular rings, valuation rings, and Prüfer/ semihereditary rings. See for instance [[8]].

Theorem 3.1 *Let R be a commutative ring and let I be a proper ideal of R. Then:*

1. *If $R \bowtie I$ is coherent, then R is coherent .*
2. *If R is a coherent ring and I is a finitely generated ideal of R, then $R \bowtie I$ is coherent.*
3. *Assume that I contains a regular element. Then $R \bowtie I$ is a coherent ring if and only if R is a coherent ring and I is a finitely generated ideal of R.*

We need the following Lemma before proving this Theorem.

Lemma 3.2 *([8, Theorem 2.4.1]). Let R be a commutative ring and let I be a proper ideal of R, then :*

1. *If R is a coherent ring and I is a finitely generated ideal of R, then R/I is a coherent ring.*
2. *If R/I is a coherent ring and I is a coherent R module, then R is a coherent ring.*

Proof of Theorem 3.1.

1. Let $L = \sum_{i=1}^{n} Ra_i$ be a finitely generated ideal of R, and set $J := \sum_{i=1}^{n}(R \bowtie I)(a_i, a_i)$. Consider the exact sequence of $R \bowtie I$ -modules:
$$0 \to ker(u) \to (R \bowtie I)^n \xrightarrow{u} J \to 0$$
where $u(r_i, r_i + e_i)_{1\leq i\leq n} = \sum_{i=1}^{n}(r_i, r_i + e_i)(a_i, a_i) = (\sum_{i=1}^{n} a_i r_i, \sum_{i=1}^{n} a_i r_i + \sum_{i=1}^{n} a_i e_i)$. Thus $ker(u) = \{(r_i, r_i + e_i)_{1\leq i\leq n} \in (R \bowtie I)^n / \sum_{i=1}^{n} r_i a_i = 0, \sum_{i=1}^{n} e_i a_i = 0\}$. On other hand , consider the exact sequence of R-modules:
$$0 \to ker(v) \to R^n \xrightarrow{v} L \to 0$$

where $v(b_i) = \sum_{i=1}^n b_i a_i$. Hence, $ker(u) = \{(r_i, r_i + e_i)_{1 \leq i \leq n} \in (R \bowtie I)^n / r_i \in ker(v); e_i \in I^n \cap ker(v)\}$. But J is a finitely presented since it is finitely generated and $R \bowtie I$ is coherent. Hence, $ker(u)$ is a finitely generated $(R \bowtie I)$-module and so $ker(v)$ is a finitely generated R -module . Therefore, L is a finitely presented ideal of R and so R is coherent.

2. Since I is a finitely generated ideal of R, then O_1 and O_2 are a finitely generated ideals of $R \bowtie I$. Hence, $R \bowtie I$ is a coherent ring by Lemma 3.2 since R is a coherent ring and $R \bowtie I/O_i \cong R$, as desired.

3. Assume that $R \bowtie I$ is a coherent ring. Then R is a coherent ring by 1). Now, we prove that I is a finitely generated ideal of R. Let m be a non zero element of I and set $c = (m, 0) \in R \bowtie I$. Then:

$$\begin{aligned}
(0 : c) &= \{(r, r+i) \in R \bowtie I / (r, r+i)(m, 0) = 0\} \\
&= \{(r, r+i) \in R \bowtie I / rm = 0\} \\
&= \{(r, r+i) \in R \bowtie I / r = 0\} \\
&= \{(0, i) \in R \bowtie I / i \in I\} \\
&= O_1.
\end{aligned}$$

Since $R \bowtie I$ is a coherent ring, then $(0 : c)$ is a finitely generated ideal of $R \bowtie I$ and so O_1 is a finitely generated ideal of $R \bowtie I$. This means that I is a finitely generated ideal of R.
Conversely if R is a coherent ring and I is a finitely generated ideal of R, then $R \bowtie I$ is a coherent ring by Lemma 3.2(2) and this completes the proof of Theorem 3.1. ■

If R is an integral domain, we obtain:

Corollary 3.3 *Let R be an integral domain and let I be a proper ideal of R . Then $R \bowtie I$ is a coherent ring if and only if R is a coherent ring and I is a finitely generated ideal of R.*

In general, $R \bowtie I$ is a coherent ring doesn't imply that I is a finitely generated of R as the following example shows:

Example 3.4 *Let R be a non-Noetherian Von Neumann regular ring and let I be a non finitely generated ideal of R (see for example [5]). Then $R \bowtie I$ is a coherent ring but I is not a finitely generated.*

Now, we are able to construct a new class of non-Noetherian rings.

Example 3.5 *Let R be a non-Noetherian coherent ring and let I be a finitely generated ideal of R. Then:*

1. *$R \bowtie I$ is a coherent ring by Theorem 3.1(2).*

2. *$R \bowtie I$ is non-Noetherian by [3, Corollary 3.3] since R is non-Noetherian.*

We recall that an R- module M is called a uniformly coherent R module, if M is a finitely generated R module and there is a map $\phi : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers, such that for every $n \in \mathbb{N}$, and any nonzero homomorphism $f : R^n \to M$, $ker(f)$ can be generated by $\phi(n)$ elements.

A ring R is called an uniformly coherent ring if R is uniformly coherent as a module over itself.

Recall that an uniformly coherent is a coherent ring (by [8, Theorem 6.1.1]). Also, there exists Noetherian rings which are not uniformly coherent (see [8, p. 191]). See for instance [[8, Chapter 6]].

Theorem 3.6 *Let R be a Noetherian ring and let I be a nilpotent ideal of R. Then R is an uniformly coherent ring if and only if $R \bowtie I$ is an uniformly coherent ring.*

We need the following Lemma before proving this Theorem.

Lemma 3.7 *Let R be a commutative ring and let I be a finitely generated ideal of R. If $R \bowtie I$ is an uniformly coherent ring then so is R.*

Proof. The ideal $O_1 := \{(0, i), i \in I\}$ is a finitely generated ideal of $R \bowtie I$ since I is a finitely generated ideal of R. Hence, $R :=\cong R \bowtie I/O_1$ is an uniformly coherent ring by [8, Corollary6-1-6], as desired. ■

Proof of Theorem 3.6.

If $R \bowtie I$ is an uniformly coherent ring, then so is R by Lemma 3.7 since R is Noetherian. Conversely, assume that R is an uniformly coherent ring. Let $\varphi : R \bowtie I \to (R \bowtie I)/O_1$ be a ring epimorphism. Since $R \bowtie I$ is Noetherian (since R is Noetherian), then $(R \bowtie I)/O_1$ is a finitely presented $R \bowtie I$ module. On other hand, O_1 is nilpotent (since I is nilpotent), and $(R \bowtie I)/O_1(:\cong R)$ is uniformly coherent. Hence, $R \bowtie I$ is an uniformly coherent ring by [8, theorem 6-1-8].

References

[1] M. D'Anna , A construction of Gorenstein rings, J. Algebra **306** (2006), no. 2, 507-519.

[2] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, Ark. Mat. **45** (2007), no. 2, 241-252.

[3] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties. J. Algebra Appl. **6** (2007), no. 3, 443-459.

[4] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960), 466-488.

[5] D. L. Costa, Parameterising families of non-Noetherian rings, Comm. Algebra, 22(1994), 3997-4011

[6] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press. Princeton (1956).

[7] S. H. Cox and R.L. Pendleton, Rings for which certain flat modules are projective, Amer.Maths. Soc, 150(1970), 139-156.

[8] S. Glaz; Commutative Coherent Rings, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).

[9] J. A. Huckaba, Commutative Coherent Rings with Zero Divizors. Marcel Dekker, New York Basel, (1988).

[10] W. Heinzer, J. Huckaba and I. Papick, m-canonical ideals in integral domains, Comm.Algebra 26(1998), 3021-3043.

[11] S. Kabbaj and N. Mahdou ; Trivial Extensions Defined by coherent-like condition, comm.Algebra . Marcel Dekker (2004) ,3937-3953

[12] S. Kabbaj and N. Mahdou ; Trivial extensions of local rings and a conjecture of Costa, Lecture Notes in Pure and Appl. Math., Vol.231, Marcel Dekker, New York, (2003), 301-312.

[13] N. Mahdou; On Costa's conjecture, Comm. Algebra, 29 (7) (2001), 2775-2785.

[14] N. Mahdou; On 2-Von Neumann regular rings, Comm. Algebra 33 (10) (2005), 3489-3496.

[15] H. R. Maimani and S. Yassemi, Zero-divisor graphs of amalgamated duplication of a ring along an ideal. J. Pure Appl. Algebra **212** (2008), no. 1, 168-174.

[16] M. Nagata, Local Rings, Interscience, New york, (1962).

[17] B. Nashier and W. Nichols, On Steinitz properties, Arch. Math, 57 (1991), 247-253.

[18] J. Rotman; An introduction to homological algebra, Press, New York, 25(1979).